Implicative algebras for noncommutative forcing (or: doing realizability without realizers)

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Introduction

- Classical realizability is a complete reformulation of the principles of Kleene realizability to take into account classical reasoning
  - Based on Griffin's discovery:

$$\text{call/cc} \ : \ ((A \Rightarrow B) \Rightarrow A) \Rightarrow A \qquad \text{(Peirce's law)}$$

- Many connections between classical realizability and Cohen forcing
  - Combination of classical realizability and forcing + Generalization to realizability algebras [Krivine '11, '12]
  - Computational analysis of Cohen forcing [M. '11]
- Fascinating model-theoretic perspectives [Krivine '12]
  - Classical realizability = noncommutative forcing ?
- **This talk:** An attempt to define an algebraic structure that subsumes forcing and classical realizability

The significance of classical realizability

- **Tarski models:** $[\![A]\!] \in \{0;1\}$
  - Interprets classical provability (correctness/completeness)
- **Intuitionistic realizability:** $[\![A]\!] \in \mathfrak{P}(\Lambda)$ [Kleene 45]
  - Interprets intuitionistic proofs
  - Independence results, in intuitionistic theories
  - Definitely incompatible with classical logic
- **Cohen forcing:** $[\![A]\!] \in \mathfrak{P}(C)$ [Cohen 63]
  - Independence results, in classical theories
    (Negation of continuum hypothesis, Solovay's axiom, etc.)
- **Boolean-valued models:** $[\![A]\!] \in \mathcal{B}$ [Scott, Solovay, Vopěnka]
- **Classical realizability** $[\![A]\!] \in \mathfrak{P}(\Lambda_c)$ [Krivine 94]
  - Interprets classical proofs
  - Generalizes Tarski models... and forcing!

Heyting/Boolean-valued models vs realizability

- In Boolean/Heyting-valued models (or in forcing):
  - conjunction interpreted as a meet/intersection...
  - universal quantification amounts to an infinitary conjunction
- In intuitionistic/classical realizability:
  - conjunction interpreted as a Cartesian product
  - universal quantification interpreted as an infinitary intersection ($\neq$ infinitary conjunction)

	$\wedge = \forall = \cap$	$\wedge = \times$, $\forall = \cap$
Int. logic	Heyting-valued models	Int. realizability
Class. logic	Boolean-valued models (Cohen forcing)	Class. realizability

Implicative algebras

Definition (Implicative algebra)

An implicative algebra is a triple $(\mathscr{A}, \leq, \to)$ where

1. $(\mathscr{A}, \leq)$ is a complete (meet semi-)lattice
2. $(\to) : \mathscr{A}^2 \to \mathscr{A}$ is a binary operation such that:

   (2a) $a' \leq a,\ b \leq b'$ entails $(a \to b) \leq (a' \to b')$ $\quad (a, a', b, b' \in \mathscr{A})$

   (2b) $\bigwedge_{b \in B} (a \to b) \ = \ a \to \bigwedge_{b \in B} b$ $\quad$ (for all $B \subseteq \mathscr{A}$)

- Write $\bot/\top$ the smallest/largest elements of $\mathscr{A}$
- When $B = \varnothing$, axiom (2b) gives: $(a \to \top) = \top$ $\quad (a \in \mathscr{A})$

Examples of implicative algebras

- **Complete Heyting algebras**

  Where $\to$ is defined as Heyting's implication:

  $$a \to b \;=\; \max\{c \in \mathscr{A} \;:\; (c \wedge a) \leq b\}$$

  so that $$(c \wedge a) \leq b \quad \text{iff} \quad c \leq (a \to b)$$

- **Complete Boolean algebras** (as particular cases of Heyting algebras)

- **Krivine's classical realizability algebras**

  Given a Krivine algebra $(\mathbf{\Lambda}, \mathbf{\Pi}, \mathbf{\Lambda} \star \mathbf{\Pi}, \ldots, \perp\!\!\!\perp)$, we let:

  - $\mathscr{A} = \mathfrak{P}(\mathbf{\Pi})$
  - $a \leq b \;\equiv\; a \supseteq b$
  - $a \to b \;\equiv\; a^{\perp\!\!\!\perp} \cdot b$

- Many other examples coming from the λ-calculus (sets of λ-terms up to $\beta\eta$-equivalence) and from linear logic (phase semantics)

Viewing truth values as (generalized) realizers (1/2)

- The Curry-Howard correspondence:

Syntax:	Proof = Program	Formula = Type
Semantics:	Realizer	Truth value

- But in most semantics, we can associate to every realizer t its principal type $[t]$, i.e. the smallest truth value containing t

$$t \ : \ A \ \ \text{(typing)} \qquad \text{iff} \qquad [t] \leq A \ \ \text{(subtyping)}$$

- Identifying t with $[t]$, we get the inclusion

$$\text{Realizers} \ \subseteq \ \text{Truth values}$$

- Moreover, we shall see that application and abstraction can be lifted at the level of truth values. Therefore:

$$\text{Truth values} \ = \ \text{Generalized realizers}$$

Viewing truth values as (generalized) realizers (2/2)

- Fundamental ideas underlying implicative algebras
  1. Operations on λ-terms can be lifted to truth values
  2. Truth values can be used as generalized realizers
- The relation $a \leq b$ may read:
  - a is a subtype as b (viewing a and b as truth values)
  - a has type b (viewing a as a realizer, b as a truth value)
  - a is more defined than b (viewing a and b as realizers)
- In particular:

$$\text{ordering of subtyping } \leq \quad = \quad \text{reverse Scott ordering } \sqsupseteq$$

- Ideas reminiscent from earlier work of:
  - F. Ruyer (2008)
  - B. Petit (2009)
  - T. Streicher (2012)
  - W. Ferrer, J. Frey, M. Guillermo, O. Malherbe (2013)

Encoding application

Let $\mathscr{A} = (\mathscr{A}, \leq, \to)$ be an implicative algebra

Definition (Application)

Given $a, b \in \mathscr{A}$, we let: $$ab \;=\; \bigwedge\{c \in \mathscr{A} \;:\; a \leq (b \to c)\}$$

- From the point of view of Scott's ordering:
$$ab \;=\; \bigsqcup\{c \in \mathscr{A} \;:\; (b \to c) \sqsubseteq a\}$$
- **Properties:**
  - Monotonicity: $a \leq a'$, $b \leq b'$ entail $ab \leq a'b'$
  - Adjunction: $ab \leq c$ iff $a \leq b \to c$
- **Remark:** In the particular case where $\mathscr{A}$ is a complete Heyting/Boolean algebra, we have: $ab = a \wedge b$

Encoding abstraction

Let $\mathscr{A} = (\mathscr{A}, \leq, \to)$ be an implicative algebra

Definition (Abstraction)

Given $f : \mathscr{A} \to \mathscr{A}$, we let: $$\lambda f \;=\; \bigwedge_{a \in \mathscr{A}} (a \to f(a))$$

- From the point of view of Scott's ordering:
$$\lambda f \;=\; \bigsqcup_{a \in \mathscr{A}} (a \to f(a))$$
- **Properties:**
  - β-rule: $(\lambda f)a \;\leq\; f(a)$
  - η-rule: $a \;\leq\; \lambda(x \mapsto ax)$

Embedding the λ-calculus (1/2)

- More generally, we associate to every closed λ-term t a truth value $(t)^{\mathscr{A}} \in \mathscr{A}$ (using encodings of application/abstraction)

- **Properties:**
  - β-**rule:** If $t \succ_\beta^* t'$, then $(t)^{\mathscr{A}} \leq (t')^{\mathscr{A}}$
  - η-**rule:** If $t \succ_\eta^* t'$, then $(t)^{\mathscr{A}} \geq (t')^{\mathscr{A}}$

  Remark: Not a denotational model of the λ-calculus!

- **Particular case:**

  When $\mathscr{A}$ is a complete Heyting/Boolean algebra, we have:

$$(t)^{\mathscr{A}} = \top \qquad \text{for every closed } \lambda\text{-term } t$$

  (In forcing, all λ-terms collapse to the top element)

Embedding the λ-calculus (2/2)

- Remarkable identities:

$$\begin{aligned}
\mathsf{I}^{\mathscr{A}} &= (\lambda x\,.\,x)^{\mathscr{A}} = \bigwedge_{a\in\mathscr{A}}(a\to a)\\
\mathsf{K}^{\mathscr{A}} &= (\lambda xy\,.\,x)^{\mathscr{A}} = \bigwedge_{a,b\in\mathscr{A}}(a\to b\to a)\\
\mathsf{S}^{\mathscr{A}} &= (\lambda xyz\,.\,xz(yz))^{\mathscr{A}} = \bigwedge_{a,b,c\in\mathscr{A}}((a\to b\to c)\to(a\to b)\to a\to c)
\end{aligned}$$

(and similarly for combinators $\mathbf{C} = \lambda xyz\,.\,xzy$ and $\mathbf{W} = \lambda xy\,.\,xyy$)

- By analogy, we let:

$$\text{cc}^{\mathscr{A}} = \bigwedge_{a,b\in\mathscr{A}}(((a\to b)\to a)\to a) \qquad \text{(Peirce's law)}$$

From this, we extend the embedding to all closed λ_c-terms

(Here: λ_c-term = λ-term with cc, without continuation constant)

Logical strength of an implicative algebra

- **Warning!** We may have $(t)^{\mathscr{A}} = \bot$ for some closed λ-term t.

  Intuitively, this means that the corresponding term is inconsistent in (the logic represented by) the implicative algebra $\mathscr{A}$

- We say that the implicative $\mathscr{A}$ is:
  - intuitionistically consistent when $(t)^{\mathscr{A}} \neq \bot$ for all closed λ-terms
  - classically consistent when $(t)^{\mathscr{A}} \neq \bot$ for all closed λ_c-terms

- **Examples:**
  - Every nontrivial complete Heyting algebra is int. consistent
  - Every nontrivial complete Boolean algebra is classically consistent
  - Every Krivine realizability structure whose pole $\perp\!\!\!\perp$ is coherent (cf [Krivine'12]) is classically consistent

Separators (1/2)

Definition (Separator)

A separator of $\mathscr{A}$ is a subset $S \subseteq \mathscr{A}$ such that:

1. If $a \in S$ and $a \leq b$, then $b \in S$ (upwards closed)
2. $\top \in S$ (nonempty)
3. If $(a \to b) \in S$ and $a \in S$, then $b \in S$ (modus ponens)
4. $\mathbf{K}^{\mathscr{A}} = (\lambda xy\,.\,x)^{\mathscr{A}} \in S$ and $\mathbf{S}^{\mathscr{A}} = (\lambda xyz\,.\,xz(yz))^{\mathscr{A}} \in S$

Moreover:

- S is proper if $\bot \notin S$
- S is classical if $\mathrm{cc}^{\mathscr{A}} \in S$

- **Properties:** If S is a separator, then:
  - S is closed under application: $a, b \in S$ entails $ab \in S$
  - $(t)^{\mathscr{A}} \in S$ for every closed λ-term t (extends to λ_c-terms if S classical)
  - In general, S is not closed under binary meets

Separators (2/2)

- **Remark:** In the particular case where $\mathscr{A}$ is a complete Heyting or Boolean algebra, a separator is the same as a filter, since in this case we have $ab = a \wedge b$ (application = binary meet)

  Separators are just a generalization of filters in the framework of implicative algebras (noncommutative case)

- Given an implicative algebra $\mathscr{A}$ we write:
  - $S_J(\mathscr{A})$ the smallest separator of $\mathscr{A}$ (intuitionistic core)
  - $S_K(\mathscr{A})$ the smallest separator of $\mathscr{A}$ containing $\mathfrak{cc}^{\mathscr{A}}$ (classical core)

  An implicative algebra $\mathscr{A}$ is intuitionistically (resp. classically) consistent iff $\bot \notin S_J(\mathscr{A})$ (resp. $\bot \notin S_K(\mathscr{A})$)

  - When $\mathscr{A}$ is a complete Heyting algebra: $S_J(\mathscr{A}) = \{\top\}$
  - When $\mathscr{A}$ is a complete Boolean algebra: $S_K(\mathscr{A}) = \{\top\}$

Interpreting 2nd-order logic

- Every implicative algebra $\mathscr{A}$ gives a model of second-order logic:
  - Individuals interpreted by elements of $\mathbb{N}$
  - Formulas interpreted by elements of $\mathscr{A}$
  - k-ary predicates interpreted by functions $F : \mathbb{N}^k \to \mathscr{A}$

Lemma (Adequacy)

If $\quad t : \phi, \quad$ then $\quad (t)^{\mathscr{A}} \leq [\![\phi]\!]$

Proof. We more generally show (by induction) that if $\Gamma \vdash t : \phi$, then for all $\mathscr{A}$-valuations ρ and for all $\mathscr{A}$-substitutions $\sigma \leq [\![\Gamma]\!]_\rho$ we have $(t[\sigma])^{\mathscr{A}} \leq [\![\phi]\!]_\rho$.

Theorem (Soundness)

1. If $\quad \mathrm{LJ2} \vdash \phi, \quad$ then $\quad [\![\phi]\!] \in S_J(\mathscr{A})$
2. If $\quad \mathrm{LK2} \vdash \phi, \quad$ then $\quad [\![\phi]\!] \in S_K(\mathscr{A})$

Proof. If $\mathrm{LJ2} \vdash \phi$, then $t : \phi$ for some closed λ-term t. Since $(t)^{\mathscr{A}} \in S_J(\mathscr{A})$ and $(t)^{\mathscr{A}} \leq [\![\phi]\!]$ (adequacy), we get $[\![\phi]\!] \in S_J(\mathscr{A})$ (by upwards closure).

Associated tripos

- An implicative algebra $\mathscr{A}$ equipped with a separator S induces a tripos $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathrm{PreOrder}$ defined as follows:
  - For each set X, $\mathcal{P}(X) := (\mathscr{A}^X, \leq_X)$, where
    $$r \leq_X s \quad \text{iff} \quad \bigwedge_{x \in X} (f(x) \to g(x)) \in S \qquad (r, s \in \mathscr{A}^X)$$
  - For each map $f : X \to Y$, the map $\mathcal{P}(f) : \mathcal{P}(Y) \to \mathcal{P}(X)$ is given by:
    $$\mathcal{P}(f)(s) = s \circ f \qquad (s \in \mathscr{A}^Y)$$
- By construction, $\mathcal{P}(X)$ is a pre-Heyting algebra for each set X (and a pre-Boolean algebra when the separator S is classical)

Theorem (Equivalence with Krivine algebras)

When the separator S is classical, the tripos induced by $\mathscr{A}$ and S is isomorphic to a particular Krivine tripos (induced by some Krivine algebra)

Building models of ZF: the problem of extensionality

- Similar difficulty occurs when constructing
  - forcing models of ZF [Cohen'63]
  - Boolean-valued models of ZF [Scott, Solovay, Vopěnka]
  - realizability models of IZF [Myhill-Friedman'73, McCarty'84]
  - classical realizability models of ZF [Krivine'00]

  which is the interpretation of the axiom of extensionality:

$$\forall x\, \forall y\, [\forall z\, (z \in x \;\Leftrightarrow\; z \in y) \;\Rightarrow\; x = y]$$

- Reason: sets cannot be given a canonical representation
  $\rightsquigarrow$ need some extensional collapse

- Most authors solve the problem in the model, when defining the interpretation of extensional equality and membership

- Krivine addresses the problem in the syntax, using a non extensional presentation of ZF called ZF_ε (= assembly language for ZF)

The theory ZF_ε

- ZF_ε distinguishes two membership relations:
  - Intensional membership $x \,\varepsilon\, y$: stronger, non extensional
  - Extensional membership $x \in y$: weaker, defined from ε
- Language:

Formulas

$$\begin{array}{rcl} \phi, \psi & ::= & x \not\varepsilon\, y \mid x \notin y \mid x \subseteq y \\ & & \mid \bot \mid \top \mid \phi \Rightarrow \psi \mid \forall x\, \phi \end{array}$$

- Abbreviations:

$$\begin{array}{rcl} \neg\phi & \equiv & \phi \Rightarrow \bot \\ \phi \wedge \psi & \equiv & \neg(\phi \Rightarrow \psi \Rightarrow \bot) \\ \phi \vee \psi & \equiv & \neg\phi \Rightarrow \neg\psi \Rightarrow \bot \\ \exists x\, \phi & \equiv & \neg\forall x\, \neg\phi \end{array} \qquad \begin{array}{rcl} x \,\varepsilon\, y & \equiv & \neg(x \not\varepsilon\, y) \\ x \in y & \equiv & \neg(x \notin y) \\ x \approx y & \equiv & x \subseteq y \wedge y \subseteq x \end{array}$$

- A formula ϕ is extensional if it does not contain the symbol $\not\varepsilon$.
  Up to coding details, extensional formulas are the formulas of ZF

The axioms of ZF_ε

Extensionality $\forall x\, \forall y\, [x \in y \;\Leftrightarrow\; (\exists z\, \varepsilon\, y)\, (x \subseteq z \wedge z \subseteq x)]$

$\forall x\, \forall y\, [x \subseteq y \;\Leftrightarrow\; (\forall z\, \varepsilon\, x)\, z \in y]$

Foundation $\forall \vec{z}\; [\forall x\, ((\forall y\, \varepsilon\, x)\phi(y, \vec{z}) \Rightarrow \phi(x, \vec{z})) \;\Rightarrow\; \forall x\, \phi(x, \vec{z})]$

Separation $\forall \vec{z}\; \forall a\, \exists b\, \forall x\, [x\, \varepsilon\, b \;\Leftrightarrow\; x\, \varepsilon\, a \wedge \phi(x, \vec{z})]$

Pairing $\forall a\, \forall b\, \exists c\, (a\, \varepsilon\, c \wedge b\, \varepsilon\, c)$

Union $\forall a\, \exists b\, (\forall x\, \varepsilon\, a)\, (\forall y\, \varepsilon\, x)\, y\, \varepsilon\, b$

Powerset $\forall a\, \exists b\, \forall x\, (\exists y\, \varepsilon\, b)\, \forall z\, [z\, \varepsilon\, y \;\Leftrightarrow\; z\, \varepsilon\, x \wedge z\, \varepsilon\, a]$

Collection $\forall \vec{z}\; \forall a\, \exists b\, (\forall x\, \varepsilon\, a)\, [\exists y\, \phi(x, y, \vec{z}) \;\Rightarrow\; (\exists y\, \varepsilon\, b)\, \phi(x, y, \vec{z})]$

Infinity $\forall \vec{z}\; \forall a\, \exists b\, [a\, \varepsilon\, b \wedge$
$(\forall x\, \varepsilon\, b)\, (\exists y\, \phi(x, y, \vec{z}) \Rightarrow (\exists y\, \varepsilon\, b)\, \phi(x, y, \vec{z}))]$

$+$ classical reasoning

Properties of ZF_ε

- Each axiom of ZF is derivable from the axioms of ZF_ε. Hence:

Proposition

ZF_ε is an extension of ZF

- But axioms of ZF_ε do not say that intensional membership $x \,\varepsilon\, y$ is stronger than extensional membership $x \in y$

$$\mathrm{ZF}_\varepsilon \nvdash \exists x \, \exists y \, (x \in y \wedge x \not\varepsilon\, y)$$

- From this we get:

Proposition (Collapsing ε to $\in$)

If $\mathrm{ZF}_\varepsilon \vdash \phi$, then $\mathrm{ZF} \vdash \phi\{\not\varepsilon := \notin\}$

Corollary (Conservativity)

ZF_ε is a conservative extension of ZF

Building the model $\mathscr{M}^{(\mathscr{A})}$ of $\mathscr{A}$-names

- The whole construction is parameterized by:
  - An arbitrary model $\mathscr{M}$ of ZF, called the ground model
  - An implicative algebra $\mathscr{A} \in \mathscr{M}$

  From these we build a generalized realizability model $\mathscr{M}^{(\mathscr{A})}$

- Construction of the model $\mathscr{M}^{(\mathscr{A})}$:
  - By induction on $\alpha \in On\,(\subseteq \mathscr{M})$, we define a set $\mathscr{M}_\alpha^{(\mathscr{A})}$ by

$$\mathscr{M}_\alpha^{(\mathscr{A})} = \bigcup_{\beta<\alpha} (\mathscr{M}_\beta^{(\mathscr{A})} \rightharpoonup \mathscr{A})$$

  - We let $\mathscr{M}^{(\mathscr{A})} = \bigcup_{\alpha \in On} \mathscr{M}_\alpha^{(\mathscr{A})}$ the (proper) class of $\mathscr{A}$-names

- Intuitively, a $\mathscr{A}$-name $u \in \mathscr{M}^{(\mathscr{A})}$ is a partial function $u : \mathscr{M}^{(\mathscr{A})} \rightharpoonup \mathscr{A}$ whose domain is a set (of $\mathscr{A}$-names)

Interpreting formulas

- Each formula ϕ of the language of ZF_ε (with parameters in $\mathscr{M}^{(\mathscr{A})}$) is interpreted as a truth value $[\![\phi]\!] \in \mathscr{A}$

$$[\![u' \not\in u]\!] = \begin{cases} u(u') & \text{if } u' \in \mathrm{dom}(u) \\ \top & \text{otherwise} \end{cases}$$

$$[\![u \subseteq v]\!] = \bigwedge_{u' \in \mathrm{dom}(u)} ([\![u' \notin v]\!] \to [\![u' \not\in u]\!])$$

$$[\![u' \notin v]\!] = \bigwedge_{v' \in \mathrm{dom}(v)} ([\![u' \subseteq v']\!] \to [\![v' \subseteq u']\!] \to [\![v' \not\in v]\!])$$

$$[\![\bot]\!] = \bot_{\mathscr{A}} \qquad [\![\phi \Rightarrow \psi]\!] = [\![\phi]\!] \to [\![\psi]\!]$$

$$[\![\top]\!] = \top_{\mathscr{A}} \qquad [\![\forall x\, \phi(x)]\!] = \bigwedge_{u \in \mathscr{M}^{(\mathscr{A})}} [\![\phi(u)]\!]$$

Soundness

Theorem (Soundness)

If $\mathrm{ZF}_\varepsilon \vdash \phi$, then $[\![\phi]\!] \in S_K(\mathscr{A})$

- **Particular case where $\mathscr{A}$ is a complete Boolean algebra:**
  - The soundness theorem reduces to the usual statement of soundness for Boolean-valued (or forcing) models of ZF:

$$\text{If} \quad \mathrm{ZF}_{(\varepsilon)} \vdash \phi, \quad \text{then} \quad [\![\phi]\!] = \top$$

- **Particular case where $\mathscr{A}$ is a Krivine algebra:**
  - The soundness theorem reduces to the usual statement of soundness for classical realizability models of ZF:

$$\text{If} \quad \mathrm{ZF}_\varepsilon \vdash \phi, \quad \text{then} \quad \theta \in [\![\phi]\!] \quad \text{for some proof-like term } \theta$$

The theory induced by a separator

- Let $\mathscr{M}^{(\mathscr{A})}$ be the generalized realizability model built from the ground model $\mathscr{M}$ of ZF and the implicative algebra $\mathscr{A}$
- Given a proper classical separator $S \subseteq \mathscr{A}$, we consider the theory

$$\mathscr{T}_S \;=\; \{\phi \text{ closed formula} : \; [\![\phi]\!] \in S\}$$

Properties of the induced theory

1. The theory $\mathscr{T}_S$ is closed under logical consequence:
   If $\phi_1, \ldots, \phi_n \in \mathscr{T}_S$ and $\phi_1, \ldots, \phi_n \vdash_{\mathsf{LK}} \phi$, then $\phi \in \mathscr{T}_S$
2. All axioms (and thus: all theorems) of ZF_ε are in $\mathscr{T}_S$
3. The theory $\mathscr{T}_s$ is consistent: $\bot \notin \mathscr{T}_S$ (since S is proper)

- **Fundamental questions:**
  - What is the structure of the set-theoretic universe described by $\mathscr{T}_S$?
  - For which formulas ϕ do we have: $\mathscr{M} \models \phi \;\Rightarrow\; \phi \in \mathscr{T}_S$?

Comparing forcing and classical realizability (1/2)

In all cases (forcing and classical realizability), the ground model $\mathscr{M}$ induces a transitive proper class $\hat{\mathscr{M}} \subseteq \mathscr{M}^{(\mathscr{A})}$

In the case of forcing (commutative case):

- The proper class $\hat{\mathscr{M}} \subseteq \mathscr{M}^{(\mathscr{A})}$ is isomorphic to $\mathscr{M}$:
  - In particular: $\mathscr{M} \models \phi$ iff $\mathscr{M}^{(\mathscr{A})} \models \phi^{\hat{\mathscr{M}}}$
  - $\hat{\mathscr{M}}$ is a transitive model of ZF inside $\mathscr{M}^{(\mathscr{A})}$
- Ordinals are the same in $\mathscr{M}$, $\hat{\mathscr{M}}$ and $\mathscr{M}^{(\mathscr{A})}$
- Axiom of Choice is preserved from $\mathscr{M}$ to $\mathscr{M}^{(\mathscr{A})}$
- In general, there are fewer cardinals in $\mathscr{M}^{(\mathscr{A})}$ than in $\hat{\mathscr{M}}$ (and $\mathscr{M}$), unless the forcing poset fulfils the countable chain condition

Comparing forcing and classical realizability (2/2)

In the case of realizability (noncommutative case):

- The proper class $\hat{\mathscr{M}} \subseteq \mathscr{M}^{(\mathscr{A})}$ is not isomorphic to $\mathscr{M}$:
  - $\mathscr{M} \models \phi$ iff $\mathscr{M}^{(\mathscr{A})} \models \phi^{\hat{\mathscr{M}}}$ only for Horn formulas ϕ
  - $\hat{\mathscr{M}} \subseteq \mathscr{M}^{(\mathscr{A})}$ is no more a (sub-)model of ZF!
  - The class $\hat{\mathscr{M}} \subseteq \mathscr{M}^{(\mathscr{A})}$ contains more integers than $\mathscr{M}$
- The proper class $\hat{\mathscr{M}} \subseteq \mathscr{M}^{(\mathscr{A})}$ acts as a Boolean-valued model of ZF over a new Boolean algebra: $\gimel 2 \in \hat{\mathscr{M}}$:

$$\mathscr{M} \models \phi \qquad \text{iff} \qquad \mathscr{M}^{(\mathscr{A})} \models [\phi]^{\hat{\mathscr{M}}}_{\gimel 2} = 1$$

  - Boolean algebra $\gimel 2 \in \hat{\mathscr{M}}$ is characteristic from realizability: it is trivial in the case of forcing (i.e. $\gimel 2 = 2$)
- We hardly understand what happens outside $\hat{\mathscr{M}}$:
  - Full Axiom of Choice fails
  - Weird/pathological properties of cardinality

Breaking Shoenfield's barrier ?

- All Boolean-valued (or forcing) models of ZF are subject to the following limitation, known as Shoenfield's barrier:

Theorem (Shoenfield)

For every Σ^1_2/Π^1_2-formula ϕ: $\mathscr{M} \models \phi$ iff $\mathscr{M}^{(\mathscr{A})} \models \phi$

- **Reason:** In the case of forcing, we always have:
  (1) $\mathscr{M}$ is a transitive model of ZF inside $\mathscr{M}^{(\mathscr{A})}$
  (2) $\mathscr{M}$ and $\mathscr{M}^{(\mathscr{A})}$ have the same ordinals
- But in classical realizability models, both properties (1) and (2) fail!
- **Hope:** Proving independence results below Shoenfield's barrier... But there is still an arithmetic barrier:

Theorem (Krivine-Miquey)

For every arithmetical formula ϕ: $\mathscr{M} \models \phi$ iff $\mathscr{M}^{(\mathscr{A})} \models \phi$

(and this in any classical/generalized realizability model $\mathscr{M}^{(\mathscr{A})}$)